THE LIMITS OF THE FABER CASTELL 2/82 N SLIDE RULE.

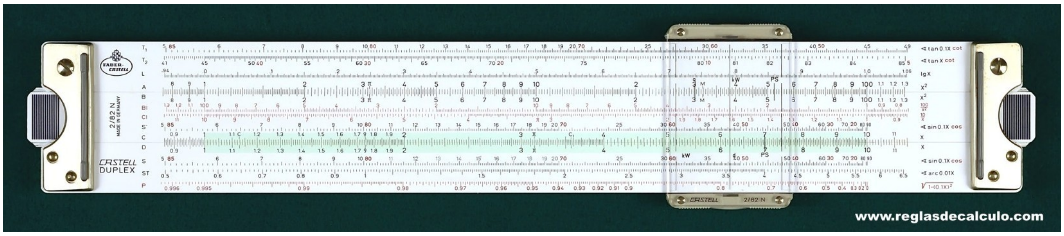

Image of a Faber Castell 2/82 N slide rule.
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The limits of the Faber Castell 2/82 N slide rule.

"It is unworthy of excellent men to waste hours like slaves in the work of calculating what could safely be relegated to someone else if machines were used." Gottfried Leibniz (1646-1716).

Contents

1 Introduction.

Since I acquired my slide rule I have been very interested in knowing what its practical limits are as a scientific calculator, that is, what can be calculated with acceptable precision.

I will briefly describe the FC 2/82 N slide rule, although I think it is quite well known. It is a 25 cm main scale slide rule, duplex type and it has all the usual scales in this type of slide rules; namely, A, B, BI, C, CI, CF, D, DF, K, K', L, LL1, LL2, LL3; LL01, LL02, LL03, P, S, S', ST, T1 and T2; that is to say, it is a fairly complete slide rule, although it is not a "big gun", as its sister is, the FC 2/83 N.

As soon as I put my hands on the slide rule, and I knew how to handle it with some ease, I considered "torturing" it a little, pushing it to the limit trying to solve some not so common problems with some numerical difficulties to see the results; and I have been dedicated to that in my free time for a couple of months until I obtained a series of conclusions that I wanted to share with those who are interested in mathematical issues a little out of the ordinary. I'm going to describe what I've been doing to finally draw some conclusions.

2 Dealing with complex numbers.

One of the the slide rule's capabilities that has surprised me is the ease to deal with complex arithmetic, I didn't expect it at all, but it's really easy. The key is to convert between Cartesian and polar forms quickly and efficiently. I will describe the method proposed in the user's manual edited by Faber Castell, that seems great to me. It is simply based on considering that for a complex number we can write:

$$z = a + bj = r\angle\theta \rightarrow \begin{cases} \tan\theta = \dfrac{b}{a} \\ \sin\theta = \dfrac{b}{r} \end{cases} \tag{1}$$

Therefore: [1]

1. If the complex number is given in Cartesian form and we want to obtain its polar form, we place b on the D scale, we align the 10 of the CI scale with the cursor, we search on the scale CI the value of a and looking at the corresponding tangent scale we have the value of θ. To do this, we have to know in advance if θ is greater or less than 45°. Well, we are actually looking for an angle in the first quadrant, which from now on I will call θ', but it is up to the calculator to know in which quadrant the polar angle really is, this will be clearly seen in the examples. With this angle θ', we have to search on the sine scale S, and aligned with it, on the CI scale we find the modulus of the complex number. All we have done are the two operations reflected above, but as we do the divisions as products by the inverse, we only have to position the slide once.

2. If the complex number is given in polar form and we want to obtain it in Cartesian form, we only have to do the inverse, that is, place θ' on the sine scale, align this angle with the modulus on the scale CI, where the 10 of it is, aligned on the scale D is b and looking at the tangent scale, aligned with the angle on the scale CI is a.

Before starting with the examples, a consideration must be made regarding the polar angle or argument.The criterion for naming angles that is preferred in mathematics is not that of the interval $[0°, 360°)$ as one might think, but rather that of $(-180°, 180°]$. This is not a whim but has to do with the fact that the complex argument is multi-evaluated (that is, we can take one or the same angle by adding or subtracting 360° as many times as we want), so we only have to take one turn, which is called the main argument. It is preferable to take the turn that goes from $-180°$ to $180°$ because thus the so-called branching cut (where the different turns overlap) corresponds to the negative real axis, not the positive one as would happen if we took the main argument between 0° and 360°. The main argument of a complex number has the following mathematical expression:

$$\mathrm{Arg}(z) = \theta = \begin{cases} \arctan\dfrac{b}{a};\ a \geq 0 \\ \arctan\dfrac{b}{a} + \mathrm{sgn}(b)\cdot 180°;\ a < 0 \end{cases} \tag{2}$$

[1] Like almost all engineers, I write the imaginary unit as j, instead of i.

The function sgn(b) means sign of b; if $b \geq 0$, it equals to 1 and 180° is added to the arctangent; if $b < 0$, it equals to -1 and 180° is subtracted from the arctangent; since the arctangent defined as a mathematical function has to be single-evaluated and therefore gives values within the interval $[-90°, 90°]$, as can be verified with any calculator, therefore with this definition of the main argument we will obtain the value in the correct quadrant given $z = a + bj$. Note that with this definition, the argument for positive reals is 0° while for negative reals it is 180°.[2]

With these previous considerations, which really are not so complicated, it is a very quick method, which with a little of practice can be mastered. And then, where is the challenge? Well, I will propose calculations that require changing between both forms repeatedly to check what happens with the precision. I will give two examples.

Example. *Calculate the expression* $\left(\frac{(7-3j)\times(-3+2j)}{-2-4j}\right)^{\frac{1}{3}}$. *Give the result in polar and Cartesian form.*

It is evident that we will have to convert the three complex numbers to polar form, operate to obtain the result and convert back to Cartesian to complete the answer. That is, several conversions and operations.

The first thing is to know in which of the two tangent scales we will look at and in which quadrant each complex number is in each case.

- $7 - 3j$ will have a value of $\theta' < 45°$; while $-90° < \theta < 0°$, that is, it is in the fourth quadrant.
- $-3 + 2j$ will have a value of $\theta' < 45°$; while $0° < \theta < 180°$, that is, it is in the second quadrant.
- $-2 - 4j$ will have a value of $\theta' > 45°$; while $-180° < \theta < -90°$, that is, it is in the third quadrant.

With these previous considerations and following the method described the result is:

$$\left(\frac{(7-3j)\times(-3+2j)}{-2-4j}\right)^{\frac{1}{3}} = \left(\frac{7.55\angle-23.2° \times 3.7\angle146.3°}{4.47\angle-116.5°}\right)^{\frac{1}{3}} \tag{3}$$

Now all we have to do is to operate as usual with complexes in polar form, that is, to multiply, the modula are multiplied and the arguments are added,

[2] This discussion that may seem artificially far-fetched is basic to maintain the coherence using complex variable functions. This is the criterion of any mathematics book that deals with the complex variable, as well as that of any calculator or spreadsheet that operates with complex numbers. Said so, there are many non-strictly mathematical books where they work with complex numbers, such as the aforementioned manual of use of the slide rule edited by Faber Castell, where the criterion is that the main argument is defined in the interval $[0°, 360°)$. You can use this last criterion as long as you are consistent, that is, you cannot mix both criteria and work at the same time with arguments of 250° and $-65°$, for example; because then we obtain absurd and erroneous results. Said so, I stick with the criterion accepted by the majority of mathematicians. A final question about notation, when we refer to the main argument we usually write Arg(z) while when we write arg(z) we are not specifying the specific turn to which we are referring.

to divide, the modula are divided and the arguments are subtracted, and to take a root, we take the root of the modulus and the argument is divided in the order of the root. Anyway, multiplying, dividing and taking a cube root are elementary operations with the slide rule and do not deserve further comment, therefore, the result is:

$$\left(\frac{7.55\angle-23.2^\circ \times 3.7\angle 146.3^\circ}{4.47\angle-116.5^\circ}\right)^{\frac{1}{3}} = (6.25\angle-120.4^\circ)^{\frac{1}{3}} = 1.84\angle-40.1^\circ \quad (4)$$

It should be noted that when we do the addition and subtraction of arguments before doing the cube root we obtain 239.6°, which we must transform into -120.4° to be consistent with the criterion of the main argument handled, since otherwise, we would have jumped between turns and the result when dividing by 3 would not be correct, because we would have mixed arguments from different turns. This is what I meant before when you have to be consistent with the criteria adopted for the main argument. This is a source of multiple mistakes when handling the complex variable and it is essential to be clear about it, you always have to move in the same direction that we have specified as the main argument, [3] If we don't take care about that, very strange things happen like this, or that the properties of logarithms do not apply, for example.

All that remains is to do the inverse procedure to obtain the result in Cartesian coordinates and we have:

$$1.84\angle-40.1^\circ = 1.42 - 1.19j \quad (5)$$

Obviously you have to pay attention to the fact that since the argument is -40.1° the number is in the fourth quadrant, which imposes the signs of each Cartesian component.

And what do a calculator tell us? Well, using my calculator, an HP 50 G that fortunately works easily with complex numbers, the result is obtained very quickly, and it turns out to be:

$$1.83\angle-40.1^\circ = 1.40 - 1.18j \quad (6)$$

I don't know what you think, but for me it is a satisfactory result, considering the number and complexity of operations involved.

Finally, to be absolutely precise, it must be said that using De Moivre's formula, the nth root of a complex number has n values of the same modulus, out of phase with each other $\dfrac{360^\circ}{n}$. In this example, since it is the cube root, there are 3 values of modulus 1.84 out of phase by 120° with each other, which form the vertices of an equilateral triangle. So:

[3] To anyone interested in delving deeper into the complex arithmetic, I would recommend any classic book on complex variable such as Markushevich's and there discover the subtleties of Riemann's surfaces.

$$(6.25\angle-120.4^\circ)^{\frac{1}{3}}=\begin{cases}1.84\angle-40.1^\circ\\1.84\angle79.9^\circ\\1.84\angle-160.1^\circ\end{cases}\tag{7}$$

Example. *Calculate the expression* $\left(\frac{4.2\angle61.2^\circ+2.7\angle-37.8^\circ}{2-3j}\right)^{\frac{1}{5}}$. *Give the result in Cartesian and polar form.*

This is a slightly more complicated example; to add you have to change to Cartesian form, then convert this sum to polar, do the division and take the fifth root, to finally pass the result again to Cartesian. I'll go directly to the results, but not before remembering again that you have to pay attention to which tangent scale you have to look at and which quadrant each complex number is in. The result is:

$$\left(\frac{4.2\angle61.2^\circ+2.7\angle-37.8^\circ}{2-3j}\right)^{\frac{1}{5}}=\left(\frac{2.02+3.68j+2.13-1.65j}{2-3j}\right)^{\frac{1}{5}}=$$
$$=\left(\frac{4.63\angle26^\circ}{3.62\angle-56.3^\circ}\right)^{\frac{1}{5}}=(1.275\angle82.3^\circ)^{\frac{1}{5}}=1.05\angle16.5^\circ=1+0.294j\tag{8}$$

Here we have to consider that to take the fifth root, we use the LL2 scale of the slide rule:

$$\sqrt[5]{1.275}=e^{\frac{1}{5}\cdot\ln 1.275}=1.05\tag{9}$$

As before, the 5 values of the fifth root are (in polar form, I propose to you as an exercise to obtain the values in Cartesian form):

$$(1.275\angle82.3^\circ)^{\frac{1}{5}}=\begin{cases}1.05\angle16.5^\circ\\1.05\angle88.5^\circ\\1.05\angle160.5^\circ\\1.05\angle-55.5^\circ\\1.05\angle-127.5^\circ\end{cases}\tag{10}$$

The calculator tells us the following result:

$$1.05\angle16.5^\circ=1+0.298j\tag{11}$$

In this case the results are even better, even with more calculations involved.

I don't think it is necessary to solve more examples, but I have done many calculations of this type and in general the results have been quite good. And so, the question is: are there situations in which this method involves some difficulty? Well, I have detected three situations in which you have to be careful with this method, two solvable and the other not so much.

2.1 Complex numbers with $0.09 < \tan\theta' < 0.1$.

T scales are intended for values $0.1 < \tan\theta' < 10$. For lower values we have to go to the ST scale and we would be in the last case that we will analyze, but..., there is a slight overlap between both scales where the T and S scales can still be used and give results with some precision, this interval is precisely $0.09 < \tan\theta' < 0.1$. If that is the case, we can still use the previous method but with caution not to make mistakes. Let me explain with an example.

Example. *Transform the complex number $z = 33 + 3j$ into polar form*

If we operate quickly and don't pay attention, we may believe that the polar angle is 42.2° because it is aligned on the scale of tangents less than 45° with the component a in the CI scale, once the 10 of the latter is aligned with the component b in the scale D. But you have to read well and understand better. The scale is labeled $\angle\tan 0.1x$, and if we look at what we have on scale D it is $x = 9.1$, which means that $0.1x = 0.91$ and not 0.091 as it should have been. Oh, decimals, how treacherous they are using the slide rule! If we have thought for a moment, we would have discovered that it is impossible for the polar angle to be 42.2° when the real component is more than 10 times the imaginary one. So, you really should have aligned with b not the 10 of the CI scale, but the 1. If we do it this way, we can use the slight overlap of the T and S scales to solve as before we have:

$$33 + 3j = 33.2\angle 5.2° \tag{12}$$

The calculator tells us a result of $33.1\angle 5.2°$, therefore the calculation with the slide rule is acceptable.

2.2 Complex numbers with $70° < \theta' < 85°$.

When the complex number we meed to convert has an argument reduced to the first quadrant $70° < \theta' < 85°$, it will be measured with low resolution on the sine scale, so if we start from the Cartesian form it will give us little precision with the modulus and if we start from the polar form we will have little precision when obtaining the Cartesian components. In this case it is better to resort to the cosine and the cotangent since:

$$z = a + bj = r\angle\theta \to \begin{cases} \cot\theta = a/b \\ \cos\theta = a/r \end{cases} \tag{13}$$

Example. *Transform the complex number $z = 1.3 + 9j$ into polar form*

If we try in this case the method explained at the beginning we see that we are dealing with an angle greater than 80°, which will give us little precision when calculating the modulus. Therefore, we simply place a on the D scale, align the 10, look in CI for the value of B and on the T1 cotangent scale (red mark, inverse scale) we read the value of 81.8°. We take this value to the S scale by looking for the cosine (red mark, inverse scale) and we have on the CI scale a modulus $r = 9.12$, that is:

$$1.3 + 9j = 9.12\angle 81.8^\circ \tag{14}$$

The calculator gives a value of $9.09\angle 81.8^\circ$, therefore I consider the calculation with the slide rule acceptable.

Example. *Transform the complex number $z = 3.5\angle 74^\circ$ into Cartesian form*

Doing the inverse procedure, we look for 74° on the cosine scale, aligning with this value on the CI scale the modulus 3.5, and under the 10 of this scale, in D, we find the value $a = 0.965$ and on the cotangent scale,looking for 74°, aligned with this value in CI we have $b = 3.38$ and so we have:

$$3.5\angle 74^\circ = 0.965 + 3.38j \tag{15}$$

The calculator gives a result of $0.965 + 3,36j$, then the result with the slide rule is quite good.

2.3 Complex numbers with $\theta' < 5^\circ$ or with $\theta' > 85^\circ$.

In these cases we are dealing with complex numbers that are either almost real or almost pure imaginary and the methods described above simply stop working since the S and T scales are merged into the ST (for sines and tangents or for cosines and cotangents, it's the same), which in practice means that the slide rule does not have the precision to distinguish between the modulus of the complex and the largest of its Cartesian components.

Example. *Transform the complex number $z = 3.5\angle 3^\circ$* into Cartesian form.

We resort to the traditional method, to do this we look at the ST scale and we have $\sin 3^\circ = 0.0525$. For the cosine we do not have a scale, but we can resort to the first term of its Taylor series, knowing when looking at the ST scale that $3^\circ = 0.0525$ rad, so $\cos 0.0525 \approx 1 - \dfrac{0.525^2}{2} = 1 - 0.00138 = 0.9986$. Therefore:

$$\begin{cases} a = 3.5\cos 3^\circ \approx 3.5 \\ b = 3.5\sin 3^\circ = 0.184 \end{cases} \tag{16}$$

As anyone can understand, there is no slide rule in the world, nor a calculator's hand that can distinguish between 1 and 0.9986; so what we can say about this complex number is that its real component is almost 3.5 and its imaginary component is 0.184, the latter being calculated with some precision. The calculator, which does not have these precision problems, tells us that:

$$3.5\angle 3^\circ = 3.495 + 0.183j \tag{17}$$

The same thing but the opposite would happen if we tried to convert to Cartesian number $z = 4.8\angle 88^\circ$.

Plainly speaking, in both cases we need more significant figures than those a slide rule can manage. This is one of the reasons why in the days of slide rules, astronomers, surveyors and others needed to resort to trigonometric and logarithm tables with many, many significant figures, because it was normal for them to deal with very little angles.

3 Calculations with slightly strange exponents.

It is common in the practice of any engineering to use experimental correlations that usually have expressions based on products and divisions of factors raised to slightly strange exponents, some of them, with a rare functional dependency as well. All these types of correlations arise from the so-called dimensional analysis when trying to propose experimental equations for models that cannot be solved analytically, based on the analysis of simpler analytical solutions. This is very common in fields such as fluid mechanics, heat transfer and many others.

All this previous considerations help me explain why I have decided to consider calculations that are not based on any of these experimental equations or correlations that I know of, but are "freely" inspired by them.

So, this section is about mixing in the same calculation, products, divisions, trigonometric functions, logarithms of various types, powers with slightly crazy exponents and that kind of things; to finally find what happens with the chained rounding errors.

Example. *Calculate the expression* $\left(\dfrac{3.28 \cdot 12\,600 \cdot 0.034}{\cos 0.4 \cdot \sqrt{215} \cdot 47}\right)^{\log_5 \sqrt[3]{\sinh 1.38}}$

First of all, a clarification, when units are not specified in a trigonometric function, it is expressed in radians, since the radian does not have dimensions because its definition is based on a geometric relationship. It is immediate by looking at the ST scale to establish that 0.4 rad = 23°. It must be said that we will use the sliding scale S' (with the caution of looking at the red scale, since we are using a cosine, not a sine) and thus there is no need to annotate intermediate results; likewise for the square root we will use scale B.

The next thing is to estimate the order of magnitude of what is inside the parentheses. I will not expand on this because it is basic use of the slide rule, I would only say that $\cos 0.4$ I estimate it to be close to 1 because it is a small angle and that $\sqrt{215}$ I estimate it to be close to 14, since $\sqrt{2} = 1.41$; considering this, and taking into account the different powers of 10, I estimate that what is inside the parentheses is worth approximately $\dfrac{12}{6} = 2$, that is enough.

The calculation of what is inside the parentheses then has no greater difficulty than making the divisions and products in the appropriate order, that is, in a zig-zag, starting with the first division so as not to have to write down intermediate results and taking advantage of the fact that on the mobile scales of the slide we have S' and B, as we have already said. Thus the result of the parentheses is:

$$\left(\frac{3.28 \cdot 12\,600 \cdot 0.034}{\cos 0.4 \cdot \sqrt{215} \cdot 47}\right) = 2.22 \tag{18}$$

It must be said that it is not common to get the estimate so close to the real result, and in fact it is not necessary, what matters is estimating correctly the order of magnitude.

To calculate the hyperbolic sine we simply place 1.38 on the D scale, and look at the values of $e^x = 3.97$ and $e^{-x} = 0.25$ on the LL3 and LL03 scales, to obtain:

$$\sinh 1.38 = \frac{3.97 - 0.25}{2} = 1.86 \tag{19}$$

To calculate the logarithm of the cube root, it is much more practical to take the logarithm and divide the result by 3; since if we place the cursor at 5 of the LL3 scale, and place the 1 of the C scale to that line, we search with the cursor for the value of 1.86, which in this case is located at the scale LL2, and in the vertical of this mark, in the C scale we find its logarithm to base 5; subsequently, we only have to divide the value obtained by 3:

$$\log_5 1.86 = 0.385 \to \log_5 \sqrt[3]{1.86} = \frac{0.385}{3} = 0.128 \tag{20}$$

And therefore we have:

$$\left(\frac{3.28 \cdot 12\,600 \cdot 0.034}{\cos 0.4 \cdot \sqrt{215} \cdot 47}\right)^{\log_5 \sqrt[3]{\sinh 1.38}} = 2.22^{0.128} \tag{21}$$

Now we only have to use the exponential scales to obtain the result without having to write down intermediate results, simply we look for $\ln 2.22$, multiply it by 0.128 and do the exponential again, taking care of what scale we use in each case, so for the first logarithm we have to use the LL2 and for the last exponential also the LL2; With these considerations we obtain:

$$2.22^{0.128} = e^{0.128 \cdot \ln 2.22} = 1.108 \tag{22}$$

The calculator gives a result of 1.106. Once again, it is surprising how much precision the slide rule can achieve with a little care and practice.

Example. *Calculate the expression* $\left(\frac{4.27 \cdot 3.28 \cdot 0.36}{\ln 4.42 \cdot \cosh 0.8 \cdot \tan 0,4}\right)^{\log(\sinh 0,7)}$.

This example is very similar, so I'm going a little faster, except in estimating what's inside the parentheses.

To estimate $\ln 4.42$ (although we can find it looking at the slide rule in a moment) we know that e is something less than 3, therefore the natural logarithm of 4 must be something greater than 1 and certainly less than 2, because e^2 is significantly larger than 4, so I estimate it to be 1.5.

To estimate the hyperbolic cosine, since it is a number less than 1, I am going to approximate it by its first term of the Taylor series, which is very similar to the cosine series, except that it is added and not subtracted. $\cosh 0.8 \approx 1 + \frac{0.8^2}{2} = 1.32$.

To estimate the tangent I do the same, $\tan 0.4 \approx 0.4$.

Thus the estimate of what is inside the parentheses is:

$$\frac{4 \cdot 3 \cdot 0.4}{1.5 \cdot 1.3 \cdot 0.4} \approx \frac{12}{2} \approx 6 \tag{23}$$

Operating as in the previous example, where the only different thing is calculating the common logarithm of a number less than 1, so care must be taken with the mantissa and the characteristic, the result is:

$$\left(\frac{4.27 \cdot 3.28 \cdot 0.36}{\ln 4.42 \cdot \cosh 0.8 \cdot \tan 0.4}\right)^{\log(\sinh 0.7)} = 6^{-0.12} = e^{-0.12 \cdot \ln 6} = 0.807 \tag{24}$$

The result using the calculator [4] is 0.807.

I think that with these two examples, although I have calculated many more, it is already clear that the slide rule performs excellently in this type of calculations. I have to say that I have been very surprised by the precision I have obtained in most of the examples of this type that I have undertaken, although many complicated operations are chained together, the precision hardly suffers.

[4] All the examples are of my invention and were not prepared a priori, that is, I had no idea what the results were when I proposed them. So, in this last example, I have to admit that I had to do the math several times to convince myself that my estimation of 6 exactly equals the result.

4 When is it better to use 10^x than e^x?

Everybody would say that it is easier and faster to handle the functions e^x and $\ln x$ than 10^x and $\log x$ when dealing with various powers and exponents because it is not necessary to write down intermediate results or take into account the characteristic and the mantissa, since the results of e^x and $\ln x$ are read directly from the slide rule.

In general this is the case, but it must be taken into account that in the slide rule the argument of the function e^x is limited to the interval $[-10, 10]$ with a slight discontinuity around $x = 0$; Furthermore, for $x > 5$ or something like that, the precision of reading the function e^x is low. This means that when dealing with large numbers things are not so clear and the function 10^x may be a better option, or a combination of both. Let's develop several examples:

Example. *Calculate* $\ln 3\,800\,000$. *Once this logarithm has been calculated, calculate its inverse to obtain the starting value again.*

We start with an easy example but it will help us establish the limits of the scales we use. Obviously if we try to look for this value on the LL scales, we are not even close, since the maximum corresponds to e^{10}, which in the rule is read as 22 000, although in reality it is 22 026.47.

If we want to use the LL scales we simply state:

$$\ln 3\,800\,000 = \ln 3.8 \cdot 10^6 = \ln 3.8 + 6 \ln 10 = 1.34 + 13.8 = 15.14 \tag{25}$$

It must be said that to do the multiplication you do not have to write down anything, just look for 10 on the LL3 scale, raise its value to the D scale and multiply by 6. Pretty easy, right? But the problem begins when doing the reverse. In effect $e^{15,14}$ goes outside the LL scales and then we have to think about something else.

The solution is to resort to the L scale, as if our slide rule were a basic one without LL scales. Thus:

$$e^{15.14} = 10^{15.14 \cdot \log e} = 10^{15.14 \cdot 0.4343} = 10^{6.58} = 10^6 \cdot 10^{0.58} = 3.8 \cdot 10^6 \tag{26}$$

Here we have obtained a good result, but we must not be fooled, we have few significant figures and the number is not too large; by the way, the value of $\log e = 0.4343$ is one of those numbers that everyone knows, if this is not the case, the L scale is used knowing that $e = 2.718\ldots$

Example. *Calculate* $2\,300^{6.4}$

First of all, we are dealing with a fairly large number, so errors when handling the functions e^x, 10^x and their corresponding logarithms can be significant. Given this fact, we adopt a strategy consisting of "cutting" the problem into independent parts, posing it as the product of an exponential with good precision and a large power of 10, thus making the errors of both operations independent and not concatenated, as will happen in any of the two strategies that we will see below, so greater precision is to be expected with this first strategy. So the approach is:

$$\begin{aligned} 2\,300^{6,4} = 2,3^{6.4} \cdot \left(10^3\right)^{6.4} = e^{6.4 \cdot \ln 2.3} \cdot 10^{19.2} = \\ = e^{6.4 \cdot \ln 2.3} \cdot 10^{0.2} \cdot 10^{19} = 210 \cdot 1.58 \cdot 10^{19} = 3.32 \cdot 10^{19} \end{aligned} \tag{27}$$

Using the calculator the result is $3.27 \cdot 10^{21}$, so we confirm that with this approach we obtain a good result, being the relative error 1.5%.

Please note that this strategy works when we have a large base (greater than 10) and a relatively small exponent, so the key is to be able to have a not very large exponential (not beyond e^6, so it can be read with acceptable precision on the slide rule). Concerning the power of 10, although it is large, it is quite precise because its exponent arises from a simple multiplication and not from a concatenation of operations, so its decimal part, which is what matters for the precision of the result, is calculated with good precision. For all these reasons, this strategy has somewhat limited validity; if the case is the opposite, that is, a base less than 10 and a large exponent, it would not apply.

If we now apply the general strategy to an exponentiation problem we write:

$$2\,300^{6.4} = e^{6.4 \cdot (\ln 2.3 + 3 \ln 10)} = e^{49.5} \tag{28}$$

We are in a similar situation to the previous example, that is, we are outside the values established in the LL scales, and in this case by a much greater margin, so we resort to a mixed strategy using first the function e^x and later 10^x and we write:

$$e^{49.5} = \left(e^{4.95}\right)^{10} = 141^{10} = 10^{10 \cdot \log 141} = 10^{10 \cdot 2.15} = 10^{21} \cdot 10^{0.5} = 3.16 \cdot 10^{21} \tag{29}$$

Given the size of numbers we handle, the calculation is not entirely bad, although it does not serve as a fine result. As expected, the relative error is greater, in this case 3.4%. This arises from concatenating the function e^x and the function $\log x$. A small error in appreciating the value of the function e^x has a significant impact on the function $\log x$ and therefore on the subsequent 10^x.

If we try the strategy of using only powers of 10 the result is:

$$2\,300^{6.4} = 10^{6.4 \cdot \log 2\,300} = 10^{6.4 \cdot 3.362} = 10^{21.5} = 10^{21} \cdot 10^{0.5} = 3.16 \cdot 10^{21} \tag{30}$$

It should be noted that in this type of calculations the difference between reading for example 21.5 on the D scale or reading 21.6 is very little, but when taking the power of 10 this difference becomes very important; so let's say that the results are not very reproducible and depend on the artist's hand.

Example. *Calculate* 2^{64}

This number is famous in mathematics for several reasons. First of all, it is the solution to the classic problem of grains of rice (or wheat, depending on versions) and chess, which is the sum of a geometric progression of ratio 2 and 64 terms, the first term being 1. Well, actually the sum of the progression is $2^{64}-1$, which is subtracting 1 grain from an ocean of rice, but, being rigorous, it serves to establish that the sum of the progression is an odd number. These types of numbers 2^n-1 are called Mersenne numbers and are used to find really large prime numbers. In fact, the largest known prime number is the Mersenne number corresponding to $n = 82\,589\,933$ and has more than 24 million digits; next to it, our "giant" $2^{64}-1$ is negligible. That's how big the big numbers that mathematicians handle are. On the other hand 2^{64} is also the number of memory addresses that a processor that we have in any home computer can handle.

Anyway, in general, the strategy of dividing the problem into the product of an exponential by a large power of 10 does not work in principle because the base is $2 < 10$, although at the end of this example we will see that since 2 is a number very particular, we can use this strategy in this case. That's why we first tried the mixed strategy and then the one based exclusively on the power of 10, so, the results are:

$$2^{64} = e^{64\cdot\ln 2} = e^{44,4} = \left(e^{4,44}\right)^{10} = 84^{10} = 10^{10\cdot\log 84} = \\ = 10^{19.24} = 10^{19}\cdot 10^{0.24} = 1.74\cdot 10^{19} \quad (31)$$

Now, the strategy based exclusively on powers of 10:

$$2^{64} = 10^{64\cdot\log 2} = 10^{19.25} = 10^{19}\cdot 10^{0.25} = 1.78\cdot 10^{19} \quad (32)$$

The calculator gives a result of $1.84\cdot 10^{19}$

In this case we have gotten closer to the real value with the second strategy than with the first, but I continue to emphasize the fact of how difficult it is to discriminate the values of the exponents with the slide rule.

Finally. as we have already said, the powers of 2 are somewhat known to anyone who has worked with computers, and therefore, it is not unreasonable to assume that it is known that $2^{16} = 65\,536$. If we do not remember all the figures, at least we can be familiar with the fact that 16 bits of color depth are 65.5 k colors; in any case, it is not too difficult to calculate 2^{16} by hand. So we can write:

$$2^{64} = \left(2^{16}\right)^4 = 6.55^4\cdot\left(10^4\right)^4 = e^{4\cdot\ln 6.55}\cdot 10^{16} = \\ = 1\,800\cdot 10^{16} = 1.8\cdot 10^{19} \quad (33)$$

We could continue with more examples, but it is already clear that when we work with really large numbers, the 25 cm slide rule does not have sufficient resolution to give acceptably precise results (although in some cases we get close to the real value, in other cases we will differ quite a lot and there is no way to predict it, since a small error in the exponent can lead to an appreciable error in the result); therefore, a larger slide rule would be needed, with a scale of 50 cm (or failing that, the 2/83 N and its scale of decimal logarithms equivalent to that of a slide rule of 50 cm), but I am not very sure either that the precision improved a lot. This is then another area where logarithm tables prevailed over calculation rules.

This fact is only mitigated in the particular case of having to raise a relatively large number to a relatively small power, so, we can divide the calculation into an exponential of reasonable precision by a power of 10 large, but of reasonably precise calculation, so the product of the two results will give an acceptable precision as we have seen.

To conclude, everything we have said about very large numbers is obviously applicable to very small (positive) numbers based on calculations with negative exponents, since a negative power is the inverse of the same positive power, which is the case discussed in this section.

5 Handling factorials.

Calculating the factorial of a number slightly larger than 10 through its definition is a long and tedious process, apart from the fact that estimating the order of magnitude of the result can be complex. On the other hand, the factorials of the natural numbers from 1 to 10 are relatively easy to calculate by hand or to remember, so it doesn't make sense to calculate them with the slide rule. The following table shows the factorials from 0 to 10:

n	0	1	2	3	4	5	6	7	8	9	10
$n!$	1	1	2	6	24	120	720	5 040	40 320	362 880	3 628 800

So, can't we do something useful with the slide rule concerning factorials? Well, yes, we will use Stirling's formula, an asymptotic approximation to the factorial. Stirling's formula states that:

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\cdot\left(\frac{n}{e}\right)^n} = 1 \tag{34}$$

From which the approximation is taken:

$$n! \approx \sqrt{2\pi n}\cdot\left(\frac{n}{e}\right)^n \tag{35}$$

Actually this approximation can be better by adding to this expression the product of an infinite series in terms of $\dfrac{1}{n}$, but that is an unnecessary and unmanageable complexity with the slide rule. To handle the factorial with the slide rule, it is better to take the logarithm of the previous expression and obtain the following practical expressions based on the natural logarithm or the common logarithm, depending on what we need:

$$\begin{aligned} \ln(n!) &= 0.92 - n + \left(n+\frac{1}{2}\right)\ln n \\ \log(n!) &= 0.4 - 0.4343n + \left(n+\frac{1}{2}\right)\log n \end{aligned} \tag{36}$$

Another non-negligible advantage of Stirling's formula is that it allows us to calculate the factorial of any positive real number defined from Euler's gamma function. Indeed, Euler's gamma function is defined as:

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt \tag{37}$$

For $x > 0$ this integral converges absolutely and it is very easily shown (integrating by parts) that:

$$\Gamma(x+1) = x\cdot\Gamma(x) \tag{38}$$

So, if n is a natural number greater than zero, we can write:

$$\Gamma(n+1) = n \cdot \Gamma(n) = n \cdot (n-1) \cdot \Gamma(n-1) = \cdots = n! \tag{39}$$

Therefore, the concept of factorial is extended to positive reals such as:

$$x! = \Gamma(x+1) \tag{40}$$

And thus, by using Stirling's formula it is as easy to approximate the value of the factorial of any positive real number as it is to approximate the value of the factorial of any natural number.

Example. *Approximate the value of* 6.7! *using Stirling's formula.*

Let's start by calculating with the natural logarithm because it is faster and more convenient than the common logarithm:

$$\ln(6.5!) = 0.92 - 6.7 + 7.2 \cdot \ln 6.7 = 7.92 \to 6.7! = e^{7.92} = 2\,750 \tag{41}$$

If we calculate with the common logarithm we have:

$$\log(6.7!) = 0.4 - 0.043 \cdot 6.7 + 7.2 \cdot \log 6.7 = 3.45 \to 6.7! = 10^{3.45} = 2\,810 \tag{42}$$

Using the factorial function from my calculator (my calculator calculates factorials of real numbers) the result is 2 770. Here we must remember that we are calculating with Stirling's formula, not with the factorial function directly (or the gamma function in this case). Stirling's formula with my calculator gives a result of 2 736. So the approximation is not too bad in this case.

Example. *Approximate the value of* 15.3! *using Stirling's formula.*

This is larger number than the last one, so we will use the common logarithm:

$$\begin{aligned} \log(15.3!) = 0.4 - 0.4343 \cdot 15.3 + 15.8 \cdot \log 15.3 = 12.46 \to \\ \to 15.3! = 10^{12.46} = 10^{12} \cdot 10^{0.46} = 2.89 \cdot 10^{12} \end{aligned} \tag{43}$$

The factorial using my calculator calculator is $2.99 \cdot 10^{12}$, while Stirling's formula using my calculator is $2.97 \cdot 10^{12}$.

Logically, the larger the arguments, the greater the errors we can make, since a small error in the exponent translates into an appreciable error in the result; so you have to be aware of the validity of this method, which boils down to having a good estimate of the result, which is sometimes sufficient and other times not, depending on the case.

6 Something about combinatorics.

Doing combinatorics calculations with the slide rule is not easy, this is one of the points where any slide rule pales before the simplest scientific calculator. It happens as with the factorial, it is really tedious to obtain any binomial coefficient that is not small, and for the small ones, we can use Pascal's triangle, which allows us to obtain binomial coefficients up to at least order 10 requiring only additions. However, we can calculate not so small binomial coefficients in some cases in which the numerator does not have too many factors and the denominator is the factorial of a small number, typically up to 10 or so.

Another strategy is to calculate the factorials using Stirling's formula, but with a not so large binomial coefficient, enormous factorials will be involved in the calculation.

Example. *Calculate the number of combinations of Spanish lottery.*

In Spanish lottery 49 numbers are put in a lottery cage, and 6 are drawn regardless of the order; so, the number of combinations that can be obtained is:

$$C_{49,6} = \binom{49}{6} = \frac{49!}{6! \cdot 43!} = \frac{49 \cdot 48 \cdot 47 \cdot 46 \cdot 45 \cdot 44}{720} \tag{44}$$

To correctly estimate the order of magnitude of the result before calculating, we pair factors of the numerator starting at the extremes and realize that $49 \cdot 44$ looks like 2 000 (it is a little greater, but it doesn't matter because it has to be less than $50^2 = 2\,500$); the same thing happens if we match $48 \cdot 45$ and if we match $47 \cdot 46$. Thus we approximate each product by 2 000 and we have in the numerator $2\,000^3 = 8 \cdot 10^9$; we divide this by approximately 700 so the result should be about $1.2 \cdot 10^7$ or so. We don't need anything else, just an order of magnitude.

Therefore, what we have to do is the calculation in cascade, starting with the first division and then, we make the first product in a normal way, from there, we alternate dividing by the inverse using the CI scale and making a normal product, so we don't need to write down any intermediate results. With these considerations the result is:

$$\frac{49 \cdot 48 \cdot 47 \cdot 46 \cdot 45 \cdot 44}{720} = 1.40 \cdot 10^7 \tag{45}$$

The calculator gives us a result of 13 983 816, so the result obtained with the slide rule is almost perfect and it has been obtained very quickly. Let's compare this method with the effort of having calculated the three factorials with Stirling's formula.

7 Conclusions.

The slide rule is a great tool for doing calculations, it is incredible that with so little you can do so much and so well. I like it because it also forces those who use it to treat mathematics with respect. In other words, I think that using the slide rule helps a lot to correctly use current calculation tools, be it a scientific calculator, or a spreadsheet, or any other software.

Furthermore, if you are careful, the precision of the results can be surprising, this is perhaps what I least expected a priori, the ability to chain together complicated calculations without significantly affecting the precision of the results.

Having said all the above, we must recognize its limitations, which for people who have to really struggle every day with moderately specialized calculations, are not few. In fact, I think that's why the process of replacing the slide rule with the electronic pocket calculator was one of the fastest in the history of technology. In a few years it went from everyone having a slide rule to its disappearance. And that undoubtedly happened for very good reasons.

Considering all the above, the main limitations of the slide rule are:

1. The precision of trigonometric functions is often not enough, this is more important when dealing with small angles.

2. The precision of logarithms and exponentiation functions is not sufficient when dealing with large numbers.

3. In general, the slide rule is not good enough for integer arithmetic; so the slide rule was a good tool for engineers, who with 3 significant figures usually have enough, but not such a good tool for those who had to handle great integers with all their figures.